Solution Set 6, 18.06 Fall '12

1. Do problem 9 from 4.4

   (a) We write q_1 and q_2 as the columns of a matrix Q:

   $$Q = \begin{pmatrix} .8 & .6 \\ -.6 & .8 \\ 0 & 0 \end{pmatrix}.$$

   We compute

   $$P = QQ^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

   Indeed it is the case that $P^2 = P$.

   (b) $(QQ^T)^2 = QQ^TQQ^T = QIQ^T = QQ^T$

2. Do problem 23 from 4.4

   We take $q_1 = (1, 0, 0)$. We will find q_2 (up to scaling) by subtracting off the projection onto q_1.

   $$q_2' = (2, 0, 3) - (2, 0, 3) \cdot q_1 = (0, 0, 3).$$

   We now scale to find q_2.

   $$q_2 = q_2'/||q_2'|| = (0, 0, 1).$$

   We find q_3 (up to scaling) by subtracting off the projection onto q_1 and q_2.

   $$q_3' = (4, 5, 6) - (4, 5, 6) \cdot q_1 - (4, 5, 6) \cdot q_2 = (0, 5, 0).$$

   We now scale to find q_3.

   $$q_3 = q_3'/||q_3'|| = (0, 1, 0).$$

   This gives $A = QR$ where

   $$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

   $$R = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{pmatrix}.$$

3. Do problem 31 from 4.4

   Each column has norm 2 so we should take $c = 1/2$ ($-1/2$ works as well).

   To project b onto the first column we simply take

   $$\frac{1}{2}(1,-1,-1,-1)\cdot b = -1.$$

   Then the projection is just $-\frac{1}{2}(1,-1,-1,-1)$.

   To project onto the plane that is spanned by the first two columns, we add the projections on to both of them:

   $$\frac{1}{2}((1,-1,-1,-1)\cdot b)\frac{1}{2}(1,-1,-1,-1)+\frac{1}{2}(-1,1,-1,-1)\cdot b)(-1,1,-1,-1) =$$

   $$\frac{1}{2}((-1,1,1,1)+(1,-1,1,1)) = (0,0,1,1).$$

4. Do problem 3 from 8.5

   The zero vector is orthogonal and has length 0.

   Alternatively, $(1,-2,0,0,0\ldots)$ is orthogonal and has length $\sqrt{5}$.

5. We begin by writing their derivatives:

   $$0, -\sin(x), \cos(x), -2\sin(2x), 2\cos(2x).$$

   Then the corresponding differentiation matrix is:

   $$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}.$$

6. Matlab problem; see Matlab solutions.

7. Do problem 3 from 5.1

   (a) False: consider the two by two case: $\det(I+I) = 4$ but $1+\det(I) = 2$.

   (b) True: $|ABC| = |AB|\cdot|C| = |A|\cdot|B|\cdot|C|$.

(c) False: consider the two by two case: $\det(4I) = 16$ but $4\det(I) = 4$.

(d) False: Take $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then $AB - BA = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ which has determinant -1.

8. Do problem 12 from 5.1

For two by two matrices A, we have

$$|cA| = c^2|A|.$$

So the actual determinant is

$$(ad - bc)/(ad - bc)^2 = 1/(ad - bc).$$

9. Do problem 25 from 5.1

Notice that two times the first row is the second row, so the rows are not independent so A is not invertible and so it has determinant 0.

10. We may perform the first step of the elimination to obtain some matrix B whose determinant is the same as that of A, whose first column is a 1 or a -1 followed by zeroes, and for which the 5×5 matrix B' obtained by removing the first row and first column has entries that are all $0, 2, -2$.

Now, we consider the cofactor expansion along the first row.

$$\det(A) = A_{1,1}C_{1,1} + A_{1,2}C_{1,2} + \cdots + A_{1,6}C_{1,6},$$

where $C_{i,j}$ is the cofactor obtained by removing the ith row and the jth column.

Then all of the terms besides the first one are zero, since by removing the first row and any column that is not the first we obtain a matrix for which the first column is all zeroes and so has determinant zero.

The first term is divisible by 32. To see this, note that $A_{1,1}C_{1,1}$ is plus or minus 1 times the determinant of B'.

Then we may factor a 2 out of B' to get that $|B'| = 2^5|B'/2| = 32|B'/2|$. But every entry of $B'/2$ is an integer, and so $|B'/2|$ is an integer (to see this, look at the permutations definition of determinant). Then 32 times an integer is divisible by 32.